Indian Statistical Institute
First Semester Exam, 2006-2007
M.Math.II Year
Graph Theory and Combinatorics

Time: 3 hrs Date:04-12-06 Max. Marks : 100

Instructor: N S N Sastry

Answer questions upto a maximum of 100 marks.

1. Define a strongly regular graph. Compute the parameters of the strongly regular graph whose set of vertices is the set of all subsets of $\{1, 2, 3, 4, 5\}$ of even cardinality and two vertices are adjacent if their symmetric difference has cardinality 4. [10]

2. Let A be the adjacency matrix of a connected, k-regular, undirected graph Γ with no loops. Let $A_0 = Id$, $A_1 = A$ and, for $r \geq 2$, let A_r denote the matrix whose rows and columns are indexed by the vertices of Γ and, for vertices x and y of Γ, the $(x, y)^{\text{th}}$-entry of A_r is the number of non back tracking paths in Γ from x to y of length r. Show that
(a) $A_1^2 = A_2 + k \cdot Id$; and
(b) for $r \geq 2, A_1A_r = A_rA_1 = A_{r+1} + (k-1)A_{r-1}$. [4 + 8]

3. Stating the facts you use precisely, compute the spectrum of the Cayley graph associated with a finite group G and a subset S of G with $S = S^{-1} = S^G$. When is it a Ramanujan graph? [12]

4. (a) Define the expanding constant of a k-regular, connected, finite graph. Show that it is at most $\sqrt{2k(k - \mu_1)}$, where μ_1 is the largest nontrivial eigenvalue of the adjacency matrix of the graph.

   (b) Show that, if $h(X_n)$ denotes the expanding constant of a cycle on n vertices, then $h(X_n) \to 0$ as $n \to \infty$. [3 + 9 + 3]

5. Define a generalized quadrangle with parameters (s, t). Construct a generalized quadrangle with parameters (2,4). Show that the collinearity graph of a generalized quadrangle with parameters (s, t) is a strongly regular graph. [3 + 6 + 4]

6. Show that any 4-arc in a projective plane of order 4 is contained in a hyperoval. Use this to determine the number of hyperovals in a projective plane of order 4. [7 + 3]

7. If L_1 and L_2 are two distinct collections of 3-subsets of $P = \{1, 2, 3, 4, 5, 6, 7\}$ such that (P, L_1) and (P, L_2) are projective planes of order 2 and $|L_1 \cap L_2| \geq 2$, then show that $g(L_1) = L_2$ for some odd permutation g of P. [10]

8. Define a 1-factor and a 1-factorization of a set of six elements. Show that a projective plane of order 4 is unique. $[2 + 2 + 10]$

9. Define a perfect linear code. Define a q-ary Hamming code. Compute its parameters. $[4 + 4 + 6]$

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